On the Uniqueness of Solutions to the Gross-Pitaevskii Hierarchy

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Abstract

In this poster, we present the work of Klainerman and Machedon “On the uniqueness to solutions of the Gross-Pitaevskii Hierarchy."

Gross-Pitaevskii Hierarchy

For $k \geq 1$, consider functions $\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) : \mathbb{R} \times \mathbb{R}^{3k} \times \mathbb{R}^{3k} \to \mathbb{C}$ such that

$$\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) = \overline{\gamma^{(k)}(t, \mathbf{x}'_k, \mathbf{x}_k)}$$

and

$$\gamma^{(k)}(t, x_{\sigma(1)}, \ldots, x_{\sigma(k)}, x'_{\sigma(1)}, \ldots, x'_{\sigma(k)}) = \gamma^{(k)}(t, x_1, \ldots, x_k, x'_1, \ldots, x'_k)$$

for any permutation σ.

The GP Hierarchy is the following many body system :

$$\begin{cases} (i\partial_t + \Delta_{\pm}^{(k)})\gamma^{(k)} = \sum_{j=1}^{k} B_{j,k+1}(\gamma^{(k+1)}) \\ \gamma^{(k)}(0, \mathbf{x}_k, \mathbf{x}'_k) = \gamma^{(k)}(\mathbf{x}_k, \mathbf{x}'_k) \end{cases}, \tag{1}$$

where $\Delta_{\pm}^{(k)} = \Delta_{\mathbf{x}_k} - \Delta_{\mathbf{x}'_k}$, and $B_{j,k+1}\gamma^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1})$ is a linear operator defined by

$$\gamma^{(k+1)}(t, \mathbf{x}_k, x_j, \mathbf{x}'_k, x_j) - \gamma^{(k+1)}(t, \mathbf{x}_k, x'_j, \mathbf{x}'_k, x'_j).$$

A special solution to the GP Hierarchy is given by

$$\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) = \prod_{j=1}^{k} \phi(t, x_j)\overline{\phi(t, x'_j)},$$

where ϕ satisfies the cubic NLS in $\mathbb{R}^3$

$$(i\partial_t + \Delta)\phi = |\phi|^2\phi, \quad \phi(0,x) = \phi_0(x) \in H^1(\mathbb{R}^3).$$

Uniqueness

Theorem 1. *Consider solutions $\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k)$ of the GP Hierarchy with zero initial conditions, which verify the space-time estimates*

$$\int_0^T \left\| R^{(k)} B_{j,k+1}\gamma^{(k+1)}(t, \cdot, \cdot) \right\|_{L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})} \leq C^k, \tag{2}$$

for some $C > 0$ and all $j \leq k \in \mathbb{N}$, where $R^{(k)} = \prod_1^k (-\Delta_{x_j})^{1/2} \cdot \prod_1^k (-\Delta_{x'_j})^{1/2}$. Then $\|R^{(k)}\gamma^{(k)}(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})} = 0$ for all k, t.

Applying Strichartz estimates, one can verify that the special solution $\prod_{j=1}^k \phi(t, x_j)\overline{\phi(t, x'_j)}$ with H^1 data obeys the space-time estimate (2) ; therefore, it is the unique solution to the GP hierarchy.

Iterated Duhamel Expansion

From zero initial data, iterating Duhamel’s formula n times, we get

$$\gamma^{(1)}(t_1, \cdot) = \int_0^{t_1} e^{i(t_1 - t_2)\Delta_{\pm}^{(1)}} \sum_{j=1}^{1} B_{j,2}(\gamma^{(2)})(t_1, \cdot)\, dt_2$$

$$= \int_0^{t_1}\int_0^{t_2} \cdots \int_0^{t_n} e^{i(t_1 - t_2)\Delta_{\pm}^{(1)}} \sum_{j=1}^{1} B_{j,2} \cdot e^{i(t_2 - t_3)\Delta_{\pm}^{(2)}} \sum_{j=1}^{2} B_{j,3} \cdots$$

$$= \int_0^{t_1}\int_0^{t_2} \cdots \int_0^{t_n} \sum_{\mu \in M_{n+1}} J(\overrightarrow{t_{n+1}}; \mu) d\vec{t}$$

where M_{n+1} is the set of all maps $\mu : \{2, \cdots n+1\} \to \{1, \cdots n\}$ such that $\mu(k) < k$ for all k and $J(\overrightarrow{t_{n+1}}; \mu)$ is the integrand corresponding to the map μ.

The overall strategy

- Obtain linear estimates on the linear operators.
- Regrouping the $n!$ integrals into classes using “combinatorial board game” such that the integral values are preserved in the same equivalence class.
- Bound the number of the classes as well as the sum of integrals in individual classes.
- Establish uniqueness over all small time intervals, then iterate.

Linear estimates

Lemma 2. *There exists a constant L, independent of j, k, such that*

$$\|R^{(k)}B_{j,k+1}(\gamma^{(k+1)})\|_{L^2(\mathbb{R} \times \mathbb{R}^{3k} \times \mathbb{R}^{3k})} \leq L\|R^{(k+1)}\gamma_0^{(k+1)}\|_{L^2(\mathbb{R}^{3(k+1)} \times \mathbb{R}^{3(k+1)})}.$$

The setup of the board game

To every integral

$$I(\mu, \sigma) = \int_{t_1 \geq t_{\sigma(2)} \cdots \geq t_{\sigma(n+1)}} e^{i(t_1 - t_2)\Delta_{\pm}^{(1)}} \mathbf{B}_{\mu(2),2} e^{i(t_2 - t_3)\Delta_{\pm}^{(2)}} \mathbf{B}_{\mu(3),3} \cdots d\vec{t}$$

we associate a “game board" of the form

$$I(\mu, \sigma) \leftrightarrow \begin{pmatrix} t_{\sigma^{-1}(2)} & t_{\sigma^{-1}(3)} & t_{\sigma^{-1}(4)} & \cdots & t_{\sigma^{-1}(n+1)} & \\ \mathbf{B}_{\mu(2),2} & B_{1,3} & \mathbf{B}_{\mu(4),4} & \cdots & \mathbf{B}_{\mu(n+1),n+1} & \text{row 1} \\ 0 & \mathbf{B}_{\mu(3),3} & B_{2,4} & \cdots & B_{2,n+1} & \text{row 2} \\ 0 & 0 & B_{3,4} & \cdots & B_{3,n+1} & \text{row 3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & B_{n,n+1} & \text{row n} \\ \text{column 2} & \text{column 3} & \text{column 4} & \cdots & \text{column n+1} & \end{pmatrix}$$

The regrouping of game boards

Our goal here is to transform any given game board via finitely many acceptable moves to a game board in an upper echelon form. An acceptable move exchanges the “cross positioned" highlighted entries in columns and rows j and $j+1$ at the same time if $\mu(j+1) < \mu(j)$. For instance,

$$\begin{pmatrix} t_2 & t_3 & t_4 & t_5 & \\ B_{1,2} & B_{1,3} & B_{1,4} & B_{1,5} & \text{R1} \\ 0 & B_{2,3} & B_{2,4} & B_{2,5} & \text{R2} \\ 0 & 0 & B_{3,4} & B_{3,5} & \text{R3} \\ 0 & 0 & 0 & B_{4,5} & \text{R4} \\ \text{C2} & \text{C3} & \text{C4} & \text{C5} & \end{pmatrix} \longrightarrow \begin{pmatrix} t_2 & t_4 & t_3 & t_5 & \\ B_{1,2} & B_{1,3} & B_{1,4} & B_{1,5} & R1 \\ 0 & B_{2,3} & B_{2,4} & B_{2,5} & R2 \\ 0 & 0 & B_{3,4} & B_{3,5} & R3 \\ 0 & 0 & 0 & B_{4,5} & R4 \\ C2 & C3 & C4 & C5 & \end{pmatrix}.$$

The importance of the acceptable move is that it transfers the integral region while preserving the integral, i.e. $I(\mu, \mathrm{id}) = I(\mu_s, \sigma)$, where μ_s is an upper echelon form.

Lemma 3. (Board Game) *Let μ_s be a special, upper echelon matrix, and write $\mu \sim \mu_s$ if μ can be reduced to μ_s in finitely many acceptable moves. There exists D_s a subset of $[0, t_1]^n$ such that*

$$\sum_{\mu \sim \mu_s} \int_0^{t_1} \cdots \int_0^{t_n} J(\overrightarrow{t_{n+1}}; \mu) d\vec{t} = \int_{D_s} J(\overrightarrow{t_{n+1}}; \mu_s) d\vec{t}$$

Here D_s is the union of all disjoint integral regions $\{t_1 \geq t_{\sigma(2)} \geq t_{\sigma(3)} \geq \cdots t_{\sigma(n+1)}\}$ for all permutations σ which occur in a given class of μ_s.

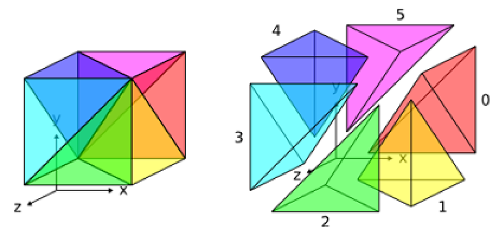
The transformation of the integral region

The essence of this Board Game is that for a given class, the acceptable moves preserve the value of integral while changing the integral regions. Thus, we can transfer integrals over the same region with different integrands to a single integral whose integral region is the disjoint union of integral regions produced by the acceptable moves.

For example, consider the case $n = 3$, $t_1 = 1$, where all possible transformed integral regions are

$$\{1 \geq t_2 \geq t_3 \geq t_4 \geq 0\}, \{1 \geq t_3 \geq t_2 \geq t_4 \geq 0\}, \cdots.$$

In general, D_s (a subset of $[0, t_1]^n$) is just the disjoint union of some n-simplices.



The counting of upper echelon matrices

We claim the following two results of crucial importance :

- For each element of M_{n+1} there is a finite set of acceptable moves which brings it to the upper echelon form.
- Let C_n be the number of $n \times n$ upper echelon matrices. Then $C_n < 4^n$.

The proof is of combinatorial nature.

Proof of Theorem 1

Applying the board game strategy (Lemma 3), we write $\gamma^{(1)}(t_1, \cdot)$ as a sum of at most 4^n terms of the form

$$\int_{D_s} J(\overrightarrow{t_{n+1}}; \mu_s) d\vec{t}. \tag{3}$$

Applying Minkowski’s inequality and commuting Fourier multipliers, we get

$$\left\| \int_{D_s} J(\overrightarrow{t_{n+1}}; \mu_s) d\vec{t} \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}$$
$$= \left\| R^{(1)} \int_{D_s} e^{i(t_1 - t_2)\Delta_{\pm}^{(1)}} B_{1,2} e^{i(t_2 - t_3)\Delta_{\pm}^{(2)}} B_{\mu_s(3),3} \cdots d\vec{t} \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}$$
$$\leq \int_{[0,t_1]^n} \| R^{(1)} B_{1,2} e^{i(t_2 - t_3)\Delta_{\pm}^{(2)}} B_{\mu_s(3),3} \cdots \|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} d\vec{t}.$$

Using the Cauchy-Schwartz inequality in t and Lemma 2 (linear estimates) $n-1$ times, we have

$$\int_{[0,t_1]^n} \| R^{(1)} B_{1,2} e^{i(t_2 - t_3)\Delta_{\pm}^{(2)}} B_{\mu_s(3),3} \cdots \|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} d\vec{t}$$
$$\leq \sqrt{t_1} \int_{[0,t_1]^{n-1}} \| R^{(1)} B_{1,2} e^{i(t_2 - t_3)\Delta_{\pm}^{(2)}} \left(B_{\mu_s(3),3} \cdots\right) \|_{L^2((t_2 \in [0,t_1]) \times \mathbb{R}^3 \times \mathbb{R}^3)} d\vec{t}$$
$$\cdots$$
$$\leq (L\sqrt{t_1})^{n-1} \int_0^{t_1} \| R^{(n)} B_{\mu_s(n+1),n+1} \gamma^{(n+1)}(t_{n+1}, \cdot) \|_{L^2(\mathbb{R}^{3n} \times \mathbb{R}^{3n})} dt_{n+1}$$
$$\leq C^n (L\sqrt{t_1})^{n-1}.$$

Consequently,

$$\| R^{(1)} \gamma^{(1)}(t_1, \cdot) \|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \leq 4^n C^n (L\sqrt{t_1})^{n-1}.$$

When $4CL\sqrt{t_1} < 1$, as $n \to \infty$, $\| R^{(1)} \gamma^{(1)}(t_1, \cdot) \|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \to 0$. Similarly, we can prove $\gamma^{(k)} = 0$ for all $k \geq 1$. Continuing this way, we get that $\gamma^{(k)} = 0$ for all $t \geq 0$, which proves the uniqueness.